MATHEMATICS

NOTES ON BANACH FUNCTION SPACES, XIII

BY

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This note is a continuation of earlier notes under the same title [5]. Some of the main results are the following.

(i) A relatively simple proof of Ogasawara's reflexivity theorem for a normed Riesz space L_ϱ (i.e., a Riesz space endowed with a Riesz norm ϱ).

(ii) Necessary and sufficient for L_ϱ to be norm complete and perfect is that ${}^0(L_{\varrho,n}^\sim)=\{0\}$ and L_ϱ has the weak Fatou property for directed sets.

(iii) If L_ϱ is norm complete and separable in the norm topology, then L_ϱ is σ-Dedekind complete if and only if $L_\varrho=L_\varrho^a$ holds.

40. *Reflexivity and perfectness*

Let L_ϱ be a normed Riesz space, i.e., a Riesz space endowed with a Riesz norm. If L_ϱ is reflexive, then L_ϱ is surely an order dense ideal in L_ϱ^{**} under the canonical imbedding of L_ϱ in L_ϱ^{**}, and hence it is necessary by Theorem 39.2 in Note XII that L_ϱ is Dedekind complete, $L_\varrho=L_\varrho^a$ and $L_\varrho^*=(L_\varrho^*)^a$. On the other hand, as shown by the space $L_\varrho=(c_0)$ with ϱ the uniform norm, these conditions are not yet sufficient for reflexivity. If L_ϱ is reflexive, then L_ϱ^{**} is surely a Dedekind completion of L_ϱ, and hence it is also necessary by Corollary 38.6 in Note XII that $0\leqslant u_\tau\uparrow$ in L_ϱ with $\sup\varrho(u_\tau)<\infty$ implies the existence of $u\in L_\varrho$ such that $u_\tau\leqslant u$ for all τ. Combining this last condition and the condition of Dedekind completeness, we obtain the condition that $0\leqslant u_\tau\uparrow$ in L_ϱ with $\sup\varrho(u_\tau)<\infty$ implies the existence of $\sup u_\tau$ in L_ϱ (the weak Fatou property for directed sets). Since $L_\varrho=L_\varrho^a$ must also hold, this can be replaced by the simpler condition that $0\leqslant u_n\uparrow$ in L_ϱ with $\sup\varrho(u_n)<\infty$ implies the existence of $\sup u_n$ in L_ϱ (the sequential weak Fatou property). The details follow below.

Theorem 40.1 (*Ogasawara's theorem*). *The following conditions are equivalent.*

(i) *L_ϱ is reflexive.*

(ii) *$L_\varrho=L_\varrho^a$, $L_\varrho^*=(L_\varrho^*)^a$, and $0\leqslant u_n\uparrow$ in L_ϱ with* $\sup\varrho(u_n)<\infty$ *implies that* $\sup u_n$ *exists in L_ϱ (weak Fatou property for sequences).*

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(iii) *$u_\tau \downarrow 0$ in L_ϱ implies $\varrho(u_\tau) \downarrow 0$; $\varphi_\tau \downarrow 0$ in L_ϱ^* implies $\varrho^*(\varphi_\tau) \downarrow 0$, and $0 \leqslant u_\tau \uparrow$ in L_ϱ with* $\sup \varrho(u_\tau) < \infty$ *implies that* $\sup u_\tau$ *exists in L_ϱ* (weak *Fatou property for directed sets).*

If L_ϱ is reflexive, then L_ϱ is super Dedekind complete and perfect.

Proof. (i) $\Rightarrow$ (ii). As already explained above, this follows by combining Theorem 39.2 and Corollary 38.6 in Note XII.

(ii) $\Rightarrow$ (iii). We have $L_\varrho = L_\varrho^a$, and $0 \leqslant u_n \uparrow$ in L_ϱ with $\sup \varrho(u_n) < \infty$ implies the existence of $\sup u_n$ in L_ϱ. Hence, by Theorem 34.2 in Note XI (the parallel to Nakano's theorem; it has come to our attention in the mean time that this parallel is essentially contained in H. Nakano's book [7], Theorem 30.20), it follows that $u_\tau \downarrow 0$ implies $\varrho(u_\tau) \downarrow 0$, and $0 \leqslant u_\tau \uparrow$ with $\sup \varrho(u_\tau) < \infty$ implies the existence of $\sup u_\tau$. Finally, observing that L_ϱ^* is Dedekind complete, we infer from $L_\varrho^* = (L_\varrho^*)^a$ by Nakano's theorem (cf. Theorem 33.4 in Note X) that $\varphi_\tau \downarrow 0$ in L_ϱ^* implies $\varrho^*(\varphi_\tau) \downarrow 0$. Alternatively, observing that L_ϱ^* has the weak Fatou property, the same conclusion may be drawn by using again the parallel theorem.

(iii) $\Rightarrow$ (i). The hypotheses in (iii) imply that L_ϱ is σ-Dedekind complete, $L_\varrho = L_\varrho^a$ and $L_\varrho^* = (L_\varrho^*)^a$. Hence, by Theorem 39.2 in Note XII, L_ϱ is an order dense ideal in L_ϱ^{**}. The hypotheses in (iii) imply also that the conditions of Corollary 38.6 in Note XII are satisfied (namely, $L_\varrho^* = L_{\varrho,n}^*$, $L_\varrho^{**} = (L_\varrho^*)_n^*$, and $0 \leqslant u_\tau \uparrow$ in L_ϱ with $\sup \varrho(u_\tau) < \infty$ implies that $u_\tau \leqslant u$ for some $u \in L_\varrho$ and all τ), and so L_ϱ^{**} is the ideal generated by L_ϱ in L_ϱ^{**}. Combining these facts, we obtain that $L_\varrho = L_\varrho^{**}$.

If L_ϱ is reflexive, then L_ϱ is σ-Dedekind complete (as already observed above). Hence, since we have also that $L_\varrho = L_\varrho^a$, it follows from Nakano's theorem (cf. Theorem 33.4 in Note X) that L_ϱ is super Dedekind complete. Finally, since any reflexive space is norm complete, it follows from Lemma 39.4 in Note XII that $L_\varrho = L_\varrho^{**} = (L_\varrho)_n^{\sim}{}_n^{\sim}$, so L_ϱ is perfect.

To the best of our knowledge, Theorem 40.1 is due to T. Ogasawara; cf. [9], Ch. V, § 4, Theorem 1 (we are indebted to Professor T. Andô for all references to this book). The book, dated 1948 and written in Japanese, evidently contains many results proved by the author in some earlier papers of 1942–1944 [8], also written in Japanese. For the special case that L_ϱ is a Banach function space, independent proofs were published in work by H. W. Ellis and I. Halperin [1], I. Halperin [3], and W. A. J. Luxemburg [4]. Ogasawara's proof uses weak and weak* sequential compactness properties of the Eberlein–Smulian type for the norm closed unit ball in L_ϱ and(or) L_ϱ^*. In comparison to this, the proof presented here seems to be more algebraic in nature.

As is well-known, not every norm complete Riesz space is reflexive. It is even so that not every norm complete space L_ϱ is perfect. By way of example, if $L_\varrho = (c_0)$, then $L_\varrho^{**} = (L_\varrho)_n^{\sim}{}_n^{\sim} = l_\infty$, so (c_0) is not perfect. It turns out, however, that the Banach dual L_ϱ^* of any normed Riesz space L_ϱ is perfect, and hence there are many norm complete Riesz spaces which are perfect but not reflexive.

Theorem 40.2. *If L_ϱ is an arbitrary normed Riesz space, then L_ϱ^* is perfect.*

Proof. For L_ϱ norm complete the proof is trivial, since in this case we have $L_\varrho^* = L_\varrho^\sim$, and $L_\varrho^\sim$ is perfect by Corollary 28.5 in Note VIII. Now assume that L_ϱ is not necessarily norm complete, and observe that the norm completion $\bar{L}_\varrho$ is a Riesz subspace of L_ϱ^{**} (cf. section 38 in Note XII). The Banach duals L_ϱ^* and $(\bar{L}_\varrho)^*$ are the same algebraically and isometrically, so if it can be shown that they are also identical with respect to the ordering the proof will be reduced to the trivial case mentioned above. All we have to show, therefore, is that L_ϱ^* and $(\bar{L}_\varrho)^*$ have the same positive elements. It is evident that if φ is a positive bounded linear functional on $\bar{L}_\varrho$, then also on L_ϱ. Conversely, if φ is a positive bounded linear functional on L_ϱ, then $u''(\varphi) \geqslant 0$ for every $0 \leqslant u'' \in L_\varrho^{**}$, so in particular for every $0 \leqslant u'' \in \bar{L}_\varrho$, and this shows that φ is a positive bounded linear functional on $\bar{L}_\varrho$.

One of the results proved in Theorem 34.3 of Note XI is that if $L_\varrho = L_\varrho^a$ and L_ϱ is perfect, then L_ϱ is norm complete if and only if the norm ϱ has the sequential weak Fatou property (i.e., if and only if $0 \leqslant u_n \uparrow$ with $\sup \varrho(u_n) < \infty$ implies the existence of $\sup u_n$ in L_ϱ). This can be improved; the hypothesis that $L_\varrho = L_\varrho^a$ is redundant. The details are as follows.

Theorem 40.3. (i) *If $0 \leqslant u_n \uparrow$ with $\sup \varrho(u_n) < \infty$ implies the existence of $\sup u_n$ in L_ϱ, then L_ϱ is norm complete.*

(ii) *If L_ϱ is perfect and norm complete, then $0 \leqslant u_\tau \uparrow$ with $\sup \varrho(u_\tau) < \infty$ implies the existence of $\sup u_\tau$ in L_ϱ.*

Proof. (i) Under the given hypothesis the space L_ϱ has the Riesz–Fischer property (if $u_n \geqslant 0$ for $n = 1, 2, \ldots$ and $\sum \varrho(u_n) < \infty$, then $\sum u_n$ exists), and so L_ϱ is norm complete by Theorem 26.3 in Note VIII.

(ii) Since L_ϱ is norm complete we have $(L_\varrho)_n^\sim = L_{\varrho,n}^*$, and hence, given $0 \leqslant u_\tau \uparrow$ in L_ϱ with $\sup \varrho(u_\tau) < \infty$, the number $p(\varphi) = \sup \varphi(u_\tau)$ is finite for every $0 \leqslant \varphi \in (L_\varrho)_n^\sim$. It follows easily that p can be linearly extended to the whole of $(L_\varrho)_n^\sim$, so p is an element of $\{(L_\varrho)_n^\sim\}^\sim$. More precisely, $p = \sup u_\tau$ in $\{(L_\varrho)_n^\sim\}^\sim$. But all the u_τ are normal integrals on $(L_\varrho)_n^\sim$, and this implies that p is also a normal integral on $(L_\varrho)_n^\sim$, so $p \in (L_\varrho)_n^\sim{}_n^\sim$. Hence, by the perfectness of L_ϱ, there exists $u \in L_\varrho$ such that $p = u$, i.e., $u = \sup u_\tau$ in L_ϱ.

Observe that $L_\varrho = l_\infty$ satisfies the hypotheses in (ii) of the present theorem, but not those of the cited Theorem 34.3 in Note XI. For the particular case that ϱ is a function norm (as introduced in Note I), the present theorem implies that if the corresponding normed function space L_ϱ is perfect as a Riesz space, then ϱ has the Riesz–Fischer property if and only if ϱ has the sequential weak Fatou property. In other words, if L_ϱ is a Banach function space (i.e., L_ϱ is norm complete), and if ϱ does not have the sequential weak Fatou property, then L_ϱ is not perfect.

There is still another aspect to part (ii) of the present Theorem 40.3. The statement in part (ii) is that if L_ϱ is perfect and norm complete, then L_ϱ has the weak Fatou property for directed sets. It may be asked now if, conversely, the weak Fatou property for directed sets in L_ϱ always implies perfectness of L_ϱ. In order to see better what this amounts to we will point out first what perfectness means in the case of a Banach function space L_ϱ. We may assume here that the point set X, on which the μ-measurable functions are defined, contains no ϱ-purely infinite sets, i.e., ϱ is a saturated function norm, and hence the first and second associates ϱ' and ϱ'' are also saturated norms (by Corollary 11.6 in Note IV). Since L_ϱ is assumed to be norm complete, we have $L_\varrho^{\sim}=L_\varrho^*$, and hence, in order to find out what $(L_\varrho)_n^{\sim}$ is, we have to determine $L_{\varrho,n}^*$. For that purpose we recall first Theorem 15.6 in Note V, where it was proved that L_ϱ is Dedekind complete. Although this was indicated only briefly, the proof shows actually that L_ϱ is super Dedekind complete (the supremum of a set of μ-measurable functions is formed by picking out an appropriate increasing subsequence). Hence $L_{\varrho,n}^*=L_{\varrho,c}^*$, and in addition (as mentioned also in section 37 of Note XII) it can be proved that $L_{\varrho,c}^*$ is exactly the first associate space L_ϱ'. This shows already that $(L_\varrho)_n^{\sim}=L_\varrho'$ algebraically, and similarly it follows then that $(L_\varrho)_{n\,n}^{\sim\sim}=L_\varrho''$ holds algebraically. Hence, perfectness of L_ϱ means that L_ϱ and L_ϱ'' contain the same functions, although $\varrho=\varrho''$ does not necessarily hold. It is easy to see, however, that in this case the norms ϱ and ϱ'' are equivalent. Elementary proof: If ϱ and ϱ'' are not equivalent, there exists a sequence $u_n\geqslant 0$ such that $\varrho''(u_n)=1$ and $\varrho(u_n)>n^3$ for all n; hence, by the norm completeness of L_ϱ'', we have $u=\sum n^{-2}u_n\in L_\varrho''=L_\varrho$. Then $\varrho''(u)<\infty$, but $\varrho(u)>n$ for every n; contradiction. Hence, if L_ϱ is a Banach function space, derived from the function norm ϱ, then L_ϱ is perfect if and only if ϱ and ϱ'' are equivalent. But, according to one of the immediate consequences of Theorem 11.4 in Note IV, ϱ and ϱ'' are equivalent if and only if ϱ has the sequential weak Fatou property. It is, therefore, not surprising that also in the case that we have an abstract normed Riesz space L_ϱ, the weak Fatou property (but now for directed sets) implies perfectness, provided at least that ${}^0(L_{\varrho,n}^{\sim})=\{0\}$, a natural condition which is automatically satisfied in the case of a function space as considered above. The proof in the abstract case requires some preliminary work which is necessary to replace, more or less, the important Lemma 11.3 of Note IV in the non-abstract case (this lemma states that $\varrho''=\varrho$ if ϱ has the Fatou property).

Finally, we observe that in the case of a function space it was not necessary to distinguish between the weak Fatou property for sequences or for directed sets, but in the abstract case this can make a difference. Indeed, considering the space in Example 29.11 of Note IX which is σ-Dedekind complete but not Dedekind complete, and choosing for ϱ the uniform norm, the thus obtained space L_ϱ has the weak Fatou property for sequences but not for directed sets. If L_ϱ is super Dedekind complete and has a countable order basis, then the weak Fatou property for sequences implies the property for directed sets. Indeed, let L_ϱ be super Dedekind complete, let $(b_n : b_{n+1} \geqslant b_n \geqslant 0;\ n=1, 2, \ldots)$ be an order basis, and let L_ϱ have the sequential weak Fatou property. Then, exactly as in Theorem 5.5 of Note II, it can be proved first that there exists a constant $k \geqslant 1$ such that $0 \leqslant u_n \uparrow u$ implies $\varrho(u) \leqslant k \sup \varrho(u_n)$. Now, let $0 \leqslant u_\tau \uparrow$ with $\sup \varrho(u_\tau) < \infty$, and set $v_{\tau,n} = \inf(u_\tau, nb_n)$. Then $s_n = \sup_\tau v_{\tau,n}$ exists by the Dedekind completeness, and by the super Dedekind completeness s_n is the supremum of a subsequence of $(v_{\tau,n} : \tau \text{ variable})$. Hence $\varrho(s_n) \leqslant k \sup \varrho(u_\tau)$ for every n. Since $s_n \uparrow$, it follows from the sequential weak Fatou property that $\sup s_n$ exists, and it is not difficult to verify that $\sup s_n = \sup u_\tau$.

41. *Perfectness and the weak Fatou property*

We start by considering the set of all Riesz seminorms on an arbitrary Riesz space L. As observed earlier, this set can be (partially) ordered by defining that $\varrho_1 \leqslant \varrho_2$ whenever $\varrho_1(f) \leqslant \varrho_2(f)$ for all $f \in L$. The set is Dedekind complete in the sense that every orderbounded subset $\{\varrho_\sigma\}$ has a supremum which satisfies $(\sup \varrho_\sigma)(f) = \sup \varrho_\sigma(f)$ for every $f \in L$. We also recall that, for ϱ_1 and ϱ_2 in the set, $\varrho_3 = \inf(\varrho_1, \varrho_2)$ exists in the set, and

$$\varrho_3(u) = \inf(\varrho_1(u') + \varrho_2(u'') : 0 \leqslant u', u'';\ u' + u'' = u)$$

for every $0 \leqslant u \in L$.

If $0 \leqslant \varphi \in L^\sim$, then $\varrho_\varphi(f) = \varphi(|f|)$ defines a Riesz seminorm ϱ_φ; for brevity we shall call ϱ_φ a *linear seminorm*. Given the Riesz seminorm ϱ and the element $0 \leqslant u \in L$, there exists by Theorem 19.2 in Note VI an element $0 \leqslant \varphi_u \in L^\sim$ such that $\varrho_{\varphi_u} \leqslant \varrho$ and $\varphi_u(u) = \varrho(u)$. Hence $\varrho = \sup_u \varrho_{\varphi_u}$, which shows that every Riesz seminorm ϱ is the supremum of all linear seminorms less than or equal to ϱ.

The Riesz seminorm ϱ will be called a *Fatou seminorm* if $0 \leqslant u_\tau \uparrow u$ implies $\varrho(u_\tau) \uparrow \varrho(u)$, and ϱ will be called a *weak Fatou seminorm* if there exists a finite constant $k(\varrho) \geqslant 1$ such that $0 \leqslant u_\tau \uparrow u$ implies

$$\varrho(u) \leqslant k(\varrho) \cdot \sup \varrho(u_\tau).$$

Finally, ϱ is called *normal* if $u_\tau \downarrow 0$ implies $\varrho(u_\tau) \downarrow 0$. Every normal semimorm is Fatou, and every Fatou seminorm is weak Fatou. If ϱ is normal and $\varrho_1 \leqslant \varrho$, then ϱ_1 is normal. Finally, if $\varrho = \sup \varrho_\sigma$ and all ϱ_σ are Fatou, then ϱ is Fatou (the proof is easy; compare Theorem 5.4 in Note II). In particular, any supremum of normal seminorms is a Fatou seminorm.

For any Riesz seminorm ϱ we define the null ideal N_ϱ by $N_\varrho = (f : f \in L,$

$\varrho(f)=0$) and the carrier C_ϱ by $C_\varrho=(N_\varrho)^p$. The carrier C_ϱ is, therefore, a normal subspace of L. If ϱ is a weak Fatou seminorm, then N_ϱ is also a normal subspace of L. If L is Dedekind complete and ϱ is weak Fatou, then $L=C_\varrho \oplus N_\varrho$, and $\varrho(u)=\varrho(u_\varrho)$ for any $0\leqslant u \in L$, where u_ϱ is the component of u in C_ϱ.

Lemma 41.1. *Let L be Dedekind complete, and let ϱ_1 and ϱ_2 be weak Fatou seminorms on L such that $C_{\varrho_2} \subset C_{\varrho_1}$. Then*

$$\sup_n \{\inf(\varrho_2, n\varrho_1)\} \leqslant \varrho_2 \leqslant k(\varrho_2)\cdot \sup_n \{\inf(\varrho_2, n\varrho_1)\}.$$

In particular, if ϱ_2 is Fatou, then $\varrho_2=\sup_n \{\inf(\varrho_2, n\varrho_1)\}$.

Proof. It is evident that the seminorm $\sup_n \{\inf(\varrho_2, n\varrho_1)\}$ exists and is less than or equal to ϱ_2. Since L is Dedekind complete, we have $L=C_{\varrho_2}\oplus N_{\varrho_2}$, and $\varrho_2(u)=\varrho_2(u_{\varrho_2})$, where u_{ϱ_2} is the component of u in C_{ϱ_2}. Hence, the inequality on the right has to be proved only on C_{ϱ_2}. Let $0\leqslant u \in C_{\varrho_2}$ and assume that $\sup_n\{\inf(\varrho_2, n\varrho_1)\}(u)<a$, so in particular $\{\inf(\varrho_2, 2^n\varrho_1)\}(u)<a$ for $n=1, 2, \ldots$. Then, for each n, there exists a decomposition $u=u_n'+u_n''$ such that $\varrho_2(u_n'')+2^n\varrho_1(u_n')<a$. By the Dedekind completeness of L, the elements $v_n=\sup(u'_{n+m} : m=0, 1, 2, \ldots)$ exist, and we have $v_n \downarrow$ and $\varrho_1(v_n)\leqslant \{k(\varrho_1)\}a/2^{n-1} \downarrow 0$. Since $v_n \in C_{\varrho_2} \subset C_{\varrho_1}$, this implies that $v_n \downarrow 0$. Then $u-v_n \leqslant u-u_n'=u_n'' \leqslant u$ with $u-v_n \uparrow u$, and so

$$\varrho_2(u)\leqslant k(\varrho_2)\cdot \sup \varrho_2(u-v_n) \leqslant k(\varrho_2)\cdot \sup \varrho_2(u_n'')<\{k(\varrho_2)\}a.$$

The desired result follows by letting $a \downarrow \sup_n \{\inf(\varrho_2, n\varrho_1)\}(u)$.

If L is not necessarily Dedekind complete but has a Dedekind completion $L^\wedge$ (i.e., if L is Archimedean), and if ϱ_1 and ϱ_2 are Fatou seminorms on L such that $C_{\varrho_2} \subset C_{\varrho_1}$, then $\varrho_2=\sup_n\{\inf(\varrho_2, n\varrho_1)\}$ still holds. In order to see this, observe first that every Fatou seminorm ϱ on L has now a unique extension $\varrho^\wedge$ to $L^\wedge$ such that $\varrho^\wedge$ is a Fatou seminorm on $L^\wedge$. Indeed, $\varrho^\wedge$ is defined for $0\leqslant u^\wedge \in L^\wedge$ by $\varrho^\wedge(u^\wedge)=\sup(\varrho(u) : u \in L, 0\leqslant u \leqslant u^\wedge)$. Also, if ϱ and λ are Fatou on L and $\varrho\leqslant\lambda$, then $\varrho^\wedge\leqslant\lambda^\wedge$. Now, let ϱ_1 and ϱ_2 be Fatou on L such that $C_{\varrho_2}\subset C_{\varrho_1}$, and let $\varrho_1^\wedge$, $\varrho_2^\wedge$ be their extensions. Since $N_{\varrho_1}\subset N_{\varrho_2}$ implies that $N_{\varrho_1^\wedge}\subset N_{\varrho_2^\wedge}$, the carrier of $\varrho_2^\wedge$ is included in the carrier of $\varrho_1^\wedge$, and hence, writing $\varrho_3^\wedge=\sup_n\{\inf(\varrho_2^\wedge, n\varrho_1^\wedge)\}$, we have $\varrho_2^\wedge=\varrho_3^\wedge$ on $L^\wedge$, which implies $\varrho_2=(\varrho_3^\wedge)_{re}$ on L, where $(\lambda^\wedge)_{re}$ denotes the restriction to L of any seminorm $\lambda^\wedge$ on $L^\wedge$. Setting now $\varrho_4=\sup_n\{\inf(\varrho_2, n\varrho_1)\}$ on L, we have evidently $\varrho_4\leqslant\varrho_2$ (note that we cannot extend ϱ_4 to $L^\wedge$ unless we prove first that ϱ_4 has Fatou; this can be proved but is not at all trivial; we shall not need it here). It is also evident that

$$\{\inf(\varrho_2^\wedge, n\varrho_1^\wedge)\}_{re}\leqslant \inf(\varrho_2, n\varrho_1),$$

and so $\sup_n\{\inf(\varrho_2^\wedge, n\varrho_1^\wedge)\}_{re}\leqslant\varrho_4$. Since we have $\sup(\lambda_\sigma^\wedge)_{re}=(\sup\lambda_\sigma^\wedge)_{re}$ for any collection of seminorms $\{\lambda_\sigma^\wedge\}$ on $L^\wedge$, it follows that $(\varrho_3^\wedge)_{re}\leqslant\varrho_4$. We had already that $\varrho_2=(\varrho_3^\wedge)_{re}$, and so we obtain now that $\varrho_2\leqslant\varrho_4$. Combining this and the earlier result that $\varrho_4 \leqslant \varrho_2$, we finally obtain $\varrho_2 = \varrho_4$, which is the desired result.

Lemma 41.2. *Let ϱ be a Riesz norm on L having the weak Fatou property for directed sets as defined in earlier sections, i.e., $0 \leqslant u_\tau \uparrow$ in L with $\sup \varrho(u_\tau) < \infty$ implies the existence of $\sup u_\tau$ in L. Then there exists a finite constant $k(\varrho) \geqslant 1$ such that $0 \leqslant u_\tau \uparrow u$ implies $\varrho(u) \leqslant k(\varrho) \cdot \sup \varrho(u_\tau)$.*

Proof. Assume there exists no such $k(\varrho)$. Then there exists for every natural number k a directed set $0 \leqslant u_{\tau k} \uparrow_\tau u_k$ such that $\varrho(u_k) > k^3 \sup_\tau \varrho(u_{\tau k})$. It is impossible that $\sup_\tau \varrho(u_{\tau k}) = 0$ for some k, for this would imply $u_{\tau k} = 0$ for all τ, and so $u_k = 0$, contradicting the inequality for $\varrho(u_k)$. Hence, multiplying by appropriate constants, we may assume that $\sup_\tau \varrho(u_{\tau k}) = k^{-2}$, and so $\varrho(u_k) > k$ for every k. Let $\{v_\sigma\}$ be the system of all finite suprema of all $u_{\tau k}$ (τ and k variable). Then $0 \leqslant v_\sigma \uparrow$ and $\varrho(v_\sigma) \leqslant \sum k^{-2} < \infty$ for every v_σ, so $v = \sup v_\sigma$ exists by hypothesis. But $v = \sup v_\sigma \geqslant \sup_\tau u_{\tau k} = u_k$ for every k, so $\varrho(v) \geqslant \varrho(u_k) \geqslant k$ for every k. Contradiction.

Lemma 41.3. *Let L_ϱ be a normed Riesz space such that ϱ has the weak Fatou property for directed sets, and denote again by $k(\varrho)$ the corresponding finite constant which exists by the preceding lemma. Furthermore, let ${}^0(L_{\varrho,n}^\sim) = \{0\}$. Then the supremum of all linear and normal seminorms, majorized by ϱ, is a norm on L_ϱ which is equivalent to ϱ. More precisely, if $\varrho'' = \sup (\varrho_\varphi : \varrho_\varphi \leqslant \varrho,\ 0 \leqslant \varphi \in L_{\varrho,n}^\sim)$, then $\varrho'' \leqslant \varrho \leqslant k^2(\varrho) \cdot \varrho''$.*

Proof. It is evident that $\varrho'' \leqslant \varrho$. For the other inequality, observe first that L_ϱ is Dedekind complete in view of ϱ having the weak Fatou property for directed sets. Let $\{\varphi_\tau\}$ be a maximal system of mutually disjoint positive and nonzero normal integrals on L_ϱ, and for each τ let A_τ be the carrier of φ_τ. On account of ${}^0(L_{\varrho,n}^\sim) = \{0\}$ the carrier of $L_{\varrho,n}^\sim$ is the whole space L_ϱ and hence, by Theorem 27.17 in Note VIII, the space L_ϱ is the Riesz direct sum of the normal subspaces A_τ. This implies that the finite direct sums of the A_τ form a directed system with respect to inclusion such that L_ϱ is the supremum of this directed system.

Let B be one of these finite direct sums. Then the corresponding finite sum of the integrals φ_τ is a strictly positive normal integral φ on B (i.e., the carrier of φ is B). The corresponding ϱ_φ is, therefore, a normal seminorm with B as its carrier, so that if we set $\varrho_B = \varrho$ on B and $\varrho_B = 0$ on the disjoint complement of B, it follows now from Lemma 41.1 that

$$\varrho_B \leqslant k(\varrho) \sup_n \{\inf (\varrho_B, n\varrho_\varphi)\}. \tag{1}$$

Now, $\inf (\varrho_B, n\varrho_\varphi)$ is normal since ϱ_φ is normal, and hence $\inf (\varrho_B, n\varrho_\varphi)$ is the supremum of a collection of normal linear seminorms, each of these seminorms being surely majorized by ϱ. The same is then true for $\sup_n \{\inf (\varrho_B, n\varrho_\varphi)\} = \varrho_B\hat{}$, and it is still true for $\varrho\hat{} = \sup_B \varrho_B\hat{}$. Hence, since for each fixed B we have $\varrho_B \leqslant k(\varrho) \cdot \varrho\hat{}$ by (1), and since $\varrho\hat{} \leqslant \varrho''$ by the just observed property of $\varrho^\wedge$ and the definition of ϱ'', we obtain already that $\varrho_B \leqslant k(\varrho)\cdot\varrho''$ for each B. To finish the proof, we note that if $0 \leqslant u \in L_\varrho$ is given and u_B is the component of u in B, then $u_B \uparrow_B u$, and so $\varrho(u) \leqslant k(\varrho)\cdot\sup_B \varrho(u_B) = k(\varrho)\cdot\sup_B \varrho_B(u) \leqslant k^2(\varrho)\cdot\varrho''(u)$.

The present proof, for the particular case that $k(\varrho)=1$, is essentially due to T. Mori, I. Amemiya and H. Nakano [6].

The main theorem on perfectness follows now.

Theorem 41.4. *For any normed Riesz space L_ϱ, the following conditions are equivalent.*

(i) *L_ϱ is norm complete and perfect.*

(ii) *L_ϱ has the weak Fatou property for directed sets (i.e., $0 \leqslant u_\tau \uparrow$ with $\sup \varrho(u_\tau) < \infty$ implies the existence of $\sup u_\tau$ in L_ϱ), and ${}^0(L^{\sim}_{\varrho,n}) = \{0\}$.*

Proof. (i) $\Rightarrow$ (ii). It was proved in Theorem 40.3 (ii) that L_ϱ has the weak Fatou property for directed sets. It is included in the definition of perfectness (cf. section 29 in Note VIII) that ${}^0(L^{\sim}_{\varrho,n}) = \{0\}$.

(ii) $\Rightarrow$ (i). It was proved in Theorem 40.3 (i) that L_ϱ is norm complete, so $L^{\sim}_\varrho = L^*_\varrho$. For the perfectness proof, observe first that L_ϱ is Dedekind complete, so that by Theorem 28.2 in Note VIII the space L_ϱ is imbedded as an ideal in $(L_\varrho)^{\sim\sim}_{nn} = (L_\varrho)^{**}_{nn}$. We have to show that $L_\varrho = (L_\varrho)^{**}_{nn}$. Considering any $0 \leqslant u \in L_\varrho$ as an element of $(L_\varrho)^{**}_{nn}$, the norm of u is

$$\sup\,(\varphi(u) : \varrho^*(\varphi) \leqslant 1,\ 0 \leqslant \varphi \in L^*_{\varrho,n}),$$

and according to the definition of ϱ'' in the preceding lemma this is exactly $\varrho''(u)$, so the preceding lemma yields that $\varrho''(u) \leqslant \varrho(u) \leqslant k^2(\varrho)\cdot\varrho''(u)$, i.e., ϱ and ϱ'' are equivalent on L_ϱ. In order to prove now that L_ϱ is perfect, it is sufficient by Theorem 28.4 in Note VIII to show that if $0 \leqslant u_\tau \uparrow$ in L_ϱ and $\sup \varphi(u_\tau) < \infty$ for every $0 \leqslant \varphi \in L^*_{\varrho,n}$, then $\sup u_\tau$ exists in L_ϱ. Given such a directed set, and considering each u_τ as a bounded linear functional on $L^*_{\varrho,n}$, we have $\sup |u_\tau(\varphi)| < \infty$ for every $\varphi \in L^*_{\varrho,n}$, so $\{\varrho''(u_\tau)\}$ is bounded by the Banach–Steinhaus theorem. But then $\{\varrho(u_\tau)\}$ is bounded by the equivalence of ϱ and ϱ'', and hence $\sup u_\tau$ exists in L_ϱ by the hypothesis that ϱ has the weak Fatou property for directed sets. This completes the proof.

Note that Theorem 40.2, i.e., the theorem that L^*_ϱ is perfect for any L_ϱ, is a very special consequence of the present theorem.

42. *Separability*

As is well-known, a topological space A is said to be separable if there exists an at most countable subset $B \subset A$ such that each open set in A contains at least one point of B. We recall that the Riesz space L is said to have a countable order basis if there exists an at most countable subset of L such that the normal subspace generated by this subset is the whole space L. Furthermore, an element $0 \leqslant u \in L$ will be called a *weak unit* of L if the normal subspace generated by u is the whole of L. In this case, therefore, the set consisting of u alone is an order basis of L.

In this section, let L_ϱ be a normed Riesz space. If we say that L_ϱ is separable, we will mean that L_ϱ is separable with respect to the norm topology.

Theorem 42.1. *If L_ϱ is separable, then L_ϱ has a countable order basis. The converse need not hold, not even if L_ϱ has the property that $u_\tau \downarrow 0$ implies $\varrho(u_\tau) \downarrow 0$. If L_ϱ is separable and norm complete, then L_ϱ has a weak unit.*

Proof. Let L_ϱ be separable, and let $\{f_n : n=1, 2, \ldots\}$ be norm dense in L_ϱ. If A is the normal subspace generated by the system $\{f_n\}$, then A is norm closed by Theorem 35.5 in Note XI, and since L_ϱ itself is the smallest norm closed linear subspace containing all the f_n, we have $A = L_\varrho$. This shows that $\{f_n\}$ is an order basis of L_ϱ. Note that $\{f_1^+, f_1^-, f_2^+, f_2^-, \ldots\}$ is also a countable order basis of L_ϱ.

Now, let L_ϱ be separable and norm complete. We may assume that L_ϱ has a countable order basis $\{u_n : n=1, 2, \ldots\}$ consisting of positive elements. Since L_ϱ is norm complete, the element

$$u = \sum_1^\infty u_n / \{n^2(\varrho(u_n)+1)\}$$

exists in L_ϱ, and the normal subspace generated by u contains all the u_n and coincides, therefore, with L_ϱ. Hence, u is a weak unit in L_ϱ. To see that the existence of a weak unit does not imply separability, not even if $u_\tau \downarrow 0$ implies $\varrho(u_\tau) \downarrow 0$, consider the case that μ is a nonseparable measure in the point set X such that $\mu(X)$ is finite, and let L_ϱ be the space $L_1(X, \mu)$ of all real μ-summable functions.

The last statement in Theorem 42.1 is due to H. Freudenthal [2].

If we want to investigate the properties of L_ϱ^* as a normed Riesz space, we may just as well assume that L_ϱ is norm complete. Indeed, if $\bar{L}_\varrho$ is the norm completion of L_ϱ, then $(\bar{L}_\varrho)^* = L_\varrho^*$, as observed in the proof of Theorem 40.2.

Theorem 42.2. *If L_ϱ is separable, then L_ϱ^* has a strictly positive bounded normal integral. Among the consequences of this fact we mention explicitly that if $0 \leqslant \varphi_\tau \uparrow \varphi$ in L_ϱ^*, then $0 \leqslant \varphi_{\tau_n} \uparrow \varphi$ for some sequence $\{\varphi_{\tau_n}\} \subset \{\varphi_\tau\}$; hence, since L_ϱ^* is Dedekind complete, L_ϱ^* is even super Dedekind complete. Furthermore, L_ϱ^* has the Egoroff property.*

Proof. As observed above, we may assume that L_ϱ is separable and norm complete. Let $u = \sum_1^\infty u_n/\{n^2(\varrho(u_n)+1)\}$ be a weak unit in L_ϱ, where $\{u_n\} = \{f_1^+, f_1^-, f_2^+, f_2^-, \ldots\}$ with $\{f_n\}$ norm dense in L_ϱ. Evidently u acts as a positive bounded normal integral on L_ϱ^*. We assert that this integral is strictly positive. Indeed, let $\varphi(u)=0$ for some $0\leqslant\varphi\in L_\varrho^*$. Then $\varphi(u_n)=0$ for all n, and so $\varphi(f_n)=0$ for all n. Since every $f\in L_\varrho$ is the norm limit of some subsequence of $\{f_n\}$, we have now that $\varphi(f)=0$ for every $f\in L_\varrho$, and so $\varphi=0$. This is the desired result. The consequences mentioned follow from Theorem 31.11 in Note X.

Before announcing the next theorem, we briefly recall some facts from the theory of normed linear spaces. Let B be a separable normed linear space, let $\{f_n : n=1, 2, \ldots\}$ be norm dense in B, and let S^* be the norm closed unit ball of the Banach dual B^*. If, for any pair $\varphi, \psi\in S^*$, we define

$$d(\varphi,\psi)=\sum_1^\infty\frac{1}{2^n}\cdot\frac{|(\varphi-\psi)f_n|}{1+|(\varphi-\psi)f_n|},$$

then this is a metric in S^*, and it is not difficult to see that every open set in this metric topology is also open in the weak* topology of S^*. Hence, the metric topology, which is a Hausdorff topology, is weaker than the weak* topology, which is a compact topology. Then the two topologies must be identical, and so S^* is a compact metric space in the weak* topology. But any compact metric space is separable, so let $\{\varphi_n : n=1, 2, \ldots\}$ be dense in the space. This means that any weak* open neighbourhood of any $\varphi\in S^*$ contains at least one of the φ_n. We assert now that if $\varphi_n(f)=0$ for some $f\in B$ and all n, then $f=0$. Indeed, given $\varphi\in S^*$, every neighbourhood $(\psi : \psi\in S^*, |(\psi-\varphi)f|<\varepsilon)$ contains at least one of the φ_n, and so $\varphi(f)=0$. But then $\varphi(f)=0$ for every $\varphi\in B^*$, so $f=0$.

Theorem 42.3. *If L_ϱ is separable, then L_ϱ has a strictly positive bounded linear functional (not necessarily an integral, however). Hence, by Theorem* 31.11 (i) *in Note X, if $0\leqslant u_\tau\uparrow u$ in L_ϱ, then $0\leqslant u_{\tau_n}\uparrow u$ for some sequence $\{u_{\tau_n}\}\subset\{u_\tau\}$ and, consequently, every integral on L_ϱ is a normal integral.*

Proof. According to the remarks above, there is a sequence $\{\varphi_n : n=1, 2, \ldots\}$ in L_ϱ^* such that $\varphi_n(f)=0$ for all n implies that $f=0$. Let

$$\varphi=\sum_1^\infty|\varphi_n|/\{n^2(\varrho^*(|\varphi_n|)+1)\}.$$

Then $0\leqslant\varphi\in L_\varrho^*$, and $\varphi(u)=0$ for $0\leqslant u\in L_\varrho$ implies that $u=0$, so φ is strictly positive on L_ϱ.

Corollary 42.4. *If L_ϱ is separable and σ-Dedekind complete, then L_ϱ is super Dedekind complete.*

Proof. Let φ be a strictly positive bounded linear functional on L_ϱ, and assume that $0\leqslant u_\tau\uparrow\leqslant v$ in L_ϱ. Let $\alpha=\sup\varphi(u_\tau)$. There exists a sequence $\{v_n=u_{\tau_n}\}\subset\{u_\tau\}$ such that $v_n\uparrow$ and $\varphi(v_n)\uparrow\alpha$. Since L_ϱ is σ-Dedekind complete, $u=\sup v_n$ exists, and all we have to prove is that u is also an upper bound of $\{u_\tau\}$. If not, there exists τ_0 such that $\sup(u_{\tau_0}, u) - u = w_0 > 0$, and so $\sup(u_{\tau_0}, v_n) - v_n \geqslant w_0 > 0$ for all n. It follows that $\varphi\{\sup(u_{\tau_0}, v_{n_0})\} > \alpha$ for some n_0. But $\{u_\tau\}$ is directed upwards, so there exists $u_{\tau_1} \geqslant \sup(u_{\tau_0}, v_{n_0})$, and hence $\varphi(u_{\tau_1}) > \alpha$. Contradiction.

One of the main theorems in this section will be that if L_ϱ is separable and norm complete, then L_ϱ is σ-Dedekind complete if and only if $L_\varrho = L_\varrho^a$ holds, i.e., if and only if every bounded linear functional on L_ϱ is an integral. The easier half is proved already in the following lemma (compare T. Ogasawara [9], Ch. III, § 7, Theorem 3).

Lemma 42.5. *If L_ϱ is separable and norm complete, and if $L_\varrho = L_\varrho^a$ holds, then L_ϱ is σ-Dedekind complete (and hence, by the preceding corollary, L_ϱ is then super Dedekind complete).*

Proof. Let $v_\tau \downarrow 0$ in L_ϱ. Then $v_{\tau_n} \downarrow 0$ for some sequence $\{v_{\tau_n}\} \subset \{v_\tau\}$ by Theorem 42.3, and so $\varrho(v_{\tau_n}) \downarrow 0$ on account of $L_\varrho = L_\varrho^a$. This shows that $v_\tau \downarrow 0$ implies $\varrho(v_\tau) \downarrow 0$, so it follows from Theorem 33.8 in Note X that every orderbounded increasing sequence in L_ϱ is a ϱ-Cauchy sequence.

Now, for the proof of σ-Dedekind completeness, let $0 \leqslant u_n \uparrow \leqslant u_0$ in L_ϱ. As observed, $\{u_n\}$ is a ϱ-Cauchy sequence, having therefore a norm limit f. But then $f = \sup u_n$ by Lemma 26.1 in Note VIII. Hence $\sup u_n$ exists, and so L_ϱ is σ-Dedekind complete.

If L is a σ-Dedekind complete Riesz space and the elements $v_n \in L$ $(n = 1, 2, \ldots)$ are pairwise disjoint and satisfy $0 \leqslant v_n \leqslant u$ for a fixed $u \in L$, then $s_n = \sum_1^n v_k$ satisfies $0 \leqslant s_n \uparrow \leqslant u$ (the inequality $s_n \leqslant u$ follows from $s_n = \sup(v_1, \ldots, v_n) \leqslant u$), so $s = \sup s_n$ exists, and $0 \leqslant s \leqslant u$. Then $r_n = \sup_k (v_n + \ldots + v_{n+k})$ exists also, and it follows from $s = s_n + r_n$ that $r_n \downarrow 0$. The same holds if L satisfies, instead of σ-Dedekind completeness, the somewhat weaker condition that, for every orderbounded sequence $\{v_n\}$ of pairwise disjoint positive elements, the element $\sup_n (v_1 + \ldots + v_n)$ exists.

Lemma 42.6. *Let L_ϱ be separable and let, for every orderbounded sequence $\{v_n\}$ of pairwise disjoint positive elements, the element* $\sup_n (v_1 + \ldots + v_n)$ *exist. Then, given $0 \leqslant v_n \leqslant u$ $(n = 1, 2, \ldots)$ in L_ϱ with the elements v_n pairwise disjoint, the elements* $r_n = \sup_k (v_n + \ldots + v_{n+k})$ *form a ϱ-Cauchy sequence.*

If L_ϱ has, in addition, the property that any ϱ-Cauchy sequence $\{u_n\}$ with $u_n \downarrow 0$ satisfies $\varrho(u_n) \downarrow 0$, then the elements r_n referred to above satisfy $\varrho(r_n) \downarrow 0$.

Proof. Assume that $\{r_n\}$ fails to be a ϱ-Cauchy sequence. Then there exists $\varepsilon > 0$ and a sequence $n_1 < n_2 < \ldots$ of indices such that $\varrho(r_{n_k} - r_{n_{k+1}}) > \varepsilon$ for all k. Hence, writing $w_k = r_{n_k} - r_{n_{k+1}}$, the elements w_k are pairwise disjoint and $\varrho(w_k) > \varepsilon$ for all k. It follows that if $\{k_i : i = 1, 2, \ldots\}$ and $\{k_j' : j = 1, 2, \ldots\}$ are arbitrary but different subsequences of the sequence of all natural numbers, the elements $t = \sup_n (w_{k_1} + \ldots + w_{k_n})$ and

$t' = \sup_n (w_{k_1'} + \dots + w_{k_n'})$ satisfy $\varrho(t-t') > \varepsilon$. The number of such t (and t') is uncountable, however, so this contradicts the separability of L_ϱ. Hence, $\{r_n\}$ is a ϱ-Cauchy sequence such that $r_n \downarrow 0$. If L_ϱ has the additional property mentioned in the statement of the theorem, then $\varrho(r_n) \downarrow 0$.

Let ϱ be a Fatou norm, i.e., $0 \leqslant u_n \uparrow u$ in L_ϱ implies $\varrho(u_n) \uparrow \varrho(u)$. Then L_ϱ has the property that every ϱ-Cauchy sequence $\{u_n\}$ with $u_n \downarrow 0$ satisfies $\varrho(u_n) \downarrow 0$. Indeed, since $u_n - u_m \uparrow_m u_n$ holds, we have then by the Fatou property that $\varrho(u_n - u_m) \uparrow \varrho(u_n)$ as $m \to \infty$. Hence, since $\varrho(u_n - u_m) \to 0$ as $m, n \to \infty$, it follows that $\varrho(u_n) \to 0$ as $n \to \infty$. For the particular case that ϱ is a Fatou norm, the present Lemma 42.6 is essentially contained in H. NAKANO's book [7], Theorem 30.27.

Theorem 42.7. *Let L_ϱ satisfy the following conditions.*

(i) *L_ϱ is separable.*

(ii) *For any orderbounded sequence $\{v_n\}$ of pairwise disjoint positive elements, the element $\sup_n (v_1 + \dots + v_n)$ exists in L_ϱ.*

(iii) *For any $0 \leqslant u, v \in L_\varrho$ the element $\sup_n \{\inf(v, nu)\}$ exists (i.e., by Corollary 29.7 in Note IX, if A_u is the principal ideal generated by u and $\{A_u\}$ is the normal subspace generated by u, then $L_\varrho = \{A_u\} \oplus A_u{}^p$).*

Then the following conditions are mutually equivalent.

(α) *$L_\varrho = L_\varrho^a$, i.e., $u_n \downarrow 0$ implies $\varrho(u_n) \downarrow 0$.*

(β) *If $\{u_n\}$ is a ϱ-Cauchy sequence such that $u_n \downarrow 0$, then $\varrho(u_n) \downarrow 0$.*

In particular, if L_ϱ is separable and σ-Dedekind complete, then (α), (β) are equivalent, and if one (and hence each) of (α), (β) holds, then L_ϱ is super Dedekind complete, and $u_\tau \downarrow 0$ implies $\varrho(u_\tau) \downarrow 0$.

Proof. We assume throughout the whole proof that (i), (ii), (iii) hold. If (α) is satisfied, then (β) is evidently satisfied. Now, assuming that (β) is satisfied, we have to show that $u \geqslant u_1 \geqslant u_2 \geqslant \dots \downarrow 0$ in L_ϱ implies $\varrho(u_n) \downarrow 0$. The proof is similar to the proof of Theorem 21.3 in Note VI (cf. also H. NAKANO [7], Theorem 30.8). For $\varrho(u) = 0$ there is nothing to prove; assume, therefore, that $\varrho(u) > 0$, and let $\{A_u\}$ be the normal subspace generated by u. Given $\varepsilon > 0$, we set $\delta = \varepsilon / \{2\varrho(u)\}$ and $p_n = (\delta u - u_n)^+$. Since $p_n \uparrow \delta u$ as $n \to \infty$, the space $\{A_u\}$ is the normal subspace generated by the sequence $\{p_n\}$, so that, denoting by u_{p_n} the component of u in the normal subspace generated by p_n, we have $u_{p_n} \uparrow u$. Indeed, assume that $u_{p_n} \uparrow \leqslant v \leqslant u$. Then $\inf(u, kp_n) = \inf(v, kp_n)$ for all k and n (as in the proof of Theorem 17.4 in Note VI), so $u_{p_n} = v_{p_n}$ for all n. But then $(u-v)_{p_n} = 0$ for all n, so $\inf(u-v, p_n) = 0$ for all n, which implies that $u - v \perp \{A_u\}$. On the other hand $u - v \in \{A_u\}$, so $u - v = 0$, i.e., $u_{p_n} \uparrow u$. Observe that, in addition, all $v_n = u_{p_n} - u_{p_{n-1}}$ are positive and pairwise disjoint. Hence, if $r_n = \sup_k (v_{n+1} + \dots + v_{n+k})$, we have $\varrho(r_n) < \varepsilon/2$ for $n \geqslant n_0$ by the preceding lemma. Observe now also that the component of $\delta u - u_n$ in the normal subspace generated by p_n is p_n itself, so the component $(u_n)_{p_n}$ of u_n is $-p_n + (\delta u)_{p_n}$. This shows that $0 \leqslant (u_n)_{p_n} \leqslant \delta u$.

The complementary component $(u_n)'_{p_n} = u_n - (u_n)_{p_n}$ satisfies $0 \leqslant (u_n)'_{p_n} \leqslant$ $\leqslant (u)'_{p_n} = r_n$. Hence

$$\varrho(u_n) \leqslant \varrho\{(u_n)_{p_n}\} + \varrho\{(u_n)'_{p_n}\} \leqslant \varrho(\delta u) + \varrho(r_n) < \varepsilon$$

for $n \geqslant n_0$. This shows that $\varrho(u_n) \downarrow 0$.

Theorem 42.8. *If L_ϱ is separable and norm complete, then $L_\varrho = L_\varrho^a$ holds if and only if L_ϱ is σ-Dedekind complete.*

Proof. Let L_ϱ be separable and norm complete. If $L_\varrho = L_\varrho^a$ holds, then L_ϱ is σ-Dedekind complete (and even super Dedekind complete) by Lemma 42.5. Conversely, if L_ϱ is σ-Dedekind complete, then conditions (i), (ii), (iii) of the preceding theorem are satisfied. Since L_ϱ is norm complete, condition (β) is also satisfied, and hence condition (α) is satisfied, i.e., $L_\varrho = L_\varrho^a$.

Corollary 42.9. *If L_ϱ is a normed Riesz space such that L_ϱ^* is separable, then $L_\varrho^* = (L_\varrho^*)^a$.*

Proof. L_ϱ^* is separable, norm complete and σ-Dedekind complete. This corollary is mentioned by T. Ogasawara [9], Ch. V, § 3, Theorem 7,

As an example to Theorem 42.8, consider the space L_ϱ of all continuous functions on $\{x : 0 \leqslant x \leqslant 1\}$ with ϱ the uniform norm. This space is separable and norm complete. The space is not σ-Dedekind complete, in accordance with the fact that $L_\varrho = L_\varrho^a$ does not hold.

With regard to Theorem 42.8 it may still be observed that in general, even when L_ϱ is norm complete and σ-Dedekind complete, the space L_ϱ is only a Riesz subspace of L_ϱ^{**} under the canonical imbedding of L_ϱ in L_ϱ^{**}. If, however, L_ϱ is separable in addition, then $L_\varrho = L_\varrho^a$ holds, and so L_ϱ is now an ideal in L_ϱ^{**} by Theorem 39.1 in Note XII.

In the next note we will discuss the connection between separability, σ-Dedekind completeness and reflexivity, and the results obtained for abstract separable Riesz spaces will be applied to separable normed function spaces. Also, a beginning will be made with extending to normal integrals the theorems proved for integrals in section 20 of Note VI and sections 24–25 of Note VII.

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